Unifying Cubical and Multimodal Type Theory

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We introduce cubical multimodal type theory ($\mathsf{MTT}_\square$), a dependent type theory that combines multimodal type theory (MTT) [GKNB20] with cubical type theory (CTT) [CCHM18]. The result not only retains the desirable qualities of both theories but also validates an extensionality principle for modal types that is not present in MTT. For semantics, we provide an axiomatic approach to constructing models, including presheaf models. As an example, we model guarded recursion by replaying the arguments from [GKNB20] now in $\mathsf{MTT}_\square$. Using presheaf models, we prove that Löb induction is consistent with the theory, and we prove using modal extensionality that Löb induction gives a propositionally unique fix-point which in MTT requires additional axioms [GKNB21].

Cubical type theory. CTT [CCHM18] was introduced to achieve two things: a computationally effective interpretation of the univalence axiom and a more well-behaved identity type satisfying e.g. function extensionality. It extends MLTT with an interval object $\mathbb{I}$—an abstraction of the interval $[0, 1]$—and *path types* $\mathsf{Path}_A(a_0, a_1)$, essentially the type of functions from the interval that agree on endpoints. A path corresponds to a term that depends on a single interval variable but dependence on multiple variables yields squares, cubes, or n-cubes. The intent is for path types to replace identity types, but they are not yet transitive. To fix this, CTT includes *Kan operations* which state that if a path is defined on only a part of an n-cube and one endpoint can be extended to the whole cube then the other endpoint may be extended as well. With this, one can prove that paths can be composed, resulting in transitivity and support for path induction (though with computation only up to a path). In order for the Kan operations to compute, they must have computation rules for every type, e.g. there is a rule specifying how the Kan operation in $A \times B$ can be reduced to operations in A and B.

Multimodal type theory. Separately, it is common to increase the expressivity of MLTT by adding *modalities* [BMSS12], but proving that these extensions satisfy desirable qualities like normalisation is laborious. MTT [GKNB20] alleviates this problem by providing a single type theory that is parametrised by a *mode theory*—a 2-category that specifies the modal situation—yet satisfies canonicity [GKNB20] and normalisation [Gra22]. By instantiating MTT with an appropriate mode theory, we can model specific modal type theories, e.g. guarded recursion [GKNB20]. Each object of a mode theory $(m, n, \ldots)$, called a mode, is a copy of MLTT, whilst each 1-cell $(\mu, \nu, \ldots)$, called a modality, allows movement between modes. Thus, for any modality $\mu : n \longrightarrow m$, we have amongst others the rules:[1]

$$\frac{\Gamma\ \mathsf{cx} \mathbin{@} m}{\Gamma, \{\mu\}\ \mathsf{cx} \mathbin{@} n} \qquad \frac{\Gamma, \{\mu\} \vdash A \mathbin{@} n}{\Gamma \vdash \langle \mu \mid A \rangle \mathbin{@} m} \qquad \frac{\Gamma, \{\mu\} \vdash a : A \mathbin{@} n}{\Gamma \vdash \mathsf{mod}_\mu(a) : \langle \mu \mid A \rangle \mathbin{@} m}$$

[1] The left-most rule is referred to as *locking* a context.

Cubical multimodal type theory. Cubical multimodal type theory ($\mathsf{MTT}_\square$) is a combination of MTT and CTT. Like MTT it is parametrised by a mode theory, but whereas MTT contains a copy of MLTT at each mode, $\mathsf{MTT}_\square$ contains a copy of CTT. Each instance of $\mathsf{MTT}_\square$ thus consists of a number of copies of CTT connected by weak dependent right adjoints.

The challenge in this combination is that in order for terms to compute, computation rules for interactions between modal and cubical aspects must be added; in particular, a computation rule for Kan operations in modal types. However, this rule will not be well-typed before adding *exchange principles*, governing interactions between the cubical and modal aspects of the theory. The exact makeup of these principles is a subtle part of the design of this theory, and care has to be taken to encapsulate the desired examples.

We adopt a principle of orthogonality, i.e. that modal and cubical aspects should interfere minimally with each other. Concretely, we have rules stating that a dimension term $\Gamma \vdash r : \mathbb{I}_m \mathbin{@} m$ may be moved to a locked context $\Gamma, \{\mu\} \vdash r^\mu : \mathbb{I}_n \mathbin{@} n$ and the same for faces. This induces substitutions $\Gamma, i : \mathbb{I}_m, \{\mu\} \vdash \sigma_\mu : \Gamma, \{\mu\}, i : \mathbb{I}_n \mathbin{@} n$ and $\Gamma, \phi, \{\mu\} \vdash \tau_\mu : \Gamma, \{\mu\}, \phi^\mu \mathbin{@} n$, where $-, \phi$ is restriction of a context to the face ϕ, which we demand are isomorphisms. A similar approach to combining CTT with a modal type theory is taken in [KMV21], whilst [Cav21] and [MV18] use equalities instead of isomorphisms.

Using these operations, we establish a computation rule for Kan operations in $\langle \mu \mid A \rangle$ in terms of A, which we prove to be well-typed; concretely:

$$\mathsf{comp}^i_{\langle \mu \mid A \rangle}\,[\phi \mapsto \mathsf{mod}_\mu(u)]\,\mathsf{mod}_\mu(u_0) = \mathsf{mod}_\mu(\mathsf{comp}^i_A\,[\phi^\mu \mapsto u[\sigma_\mu \circ \tau_\mu]]\,u_0)$$

Just as CTT validates many extensionality principles, with these exchange principles, we get for $\mathsf{MTT}_\square$ modal extensionality:

Theorem 1. *Given a modality $\mu : n \longrightarrow m$ and terms $A : \mathsf{U} \mathbin{@} n$ and $a, b : \mathsf{El}(A) \mathbin{@} n$, there is a path equivalence $\langle \mu \mid \mathsf{Path}_{\mathsf{El}(A)}(a, b) \rangle \simeq \mathsf{Path}_{\langle \mu \mid \mathsf{El}(A) \rangle}(\mathsf{mod}_\mu(a), \mathsf{mod}_\mu(b)) \mathbin{@} m$.*

$\mathsf{MTT}_\square$ is formally defined as a *generalised algebraic theory*, and it, therefore, induces a category of models, including an initial model. Due to the complexity of the type theory, constructing such a model is a laborious task, and we, therefore, introduce *cubical MTT cosmoi* as an axiomatic approach to producing models. These assign a topos satisfying axioms from [OP18] and [LOPS18] to each mode while each modality is assigned to an adjunction which induces a dependent right adjoint whose left adjoint preserves the cubical structure coherently. These axioms ensure that each mode models CTT while the entire structure models MTT. Finally, by requiring that the cubical structure is appropriately preserved these models validate the aforementioned exchange principles.

Theorem 2. *Any cubical MTT cosmos induces a model of $\mathsf{MTT}_\square$.*

Theorem 3. *Let $f : \mathcal{M} \longrightarrow$ **Cat** be a strict 2-functor, and write $F^*(\mu)$, $F_!(\mu)$, and $F_*(\mu)$ for the precomposition, left Kan extension, and right Kan extension respectively of $f(\mu) \times \mathsf{id}_\square$. Then:*

- *the network of morphisms of LOPS topoi given by the adjunctions $F^*(\mu) \dashv F_*(\mu)$ induces a model of $\mathsf{MTT}_\square$ over $\mathcal{M}$; and*
- *the network of morphisms of LOPS topoi given by the adjunctions $F_!(\mu) \dashv F^*(\mu)$ induces a model of $\mathsf{MTT}_\square$ over $\mathcal{M}$.*

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