Classifying topoi in synthetic guarded domain theory

the universal property of multi-clock guarded recursion

Daniele Palombi[a] Jonathan Sterling[b]

[a] *Department of Mathematics*
Sapienza University of Rome

[b] *Department of Computer Science*
Aarhus University

Abstract

Several different topoi have played an important role in the development and applications of *synthetic guarded domain theory* (SGDT), a new kind of synthetic domain theory that abstracts the concept of *guarded recursion* frequently employed in the semantics of programming languages. In order to unify the accounts of guarded recursion and coinduction, several authors have enriched SGDT with multiple "clocks" parameterizing different time-streams, leading to more complex and difficult to understand topos models. Until now these topoi have been understood very concretely *qua* categories of presheaves, and the logico-geometrical question of what theories these topoi classify has remained open. We show that several important topos models of SGDT classify very simple geometric theories, and that the passage to various forms of multi-clock guarded recursion can be rephrased more compositionally in terms of the lower bagtopos construction of Vickers and variations thereon due to Johnstone. We contribute to the consolidation of SGDT by isolating the universal property of multi-clock guarded recursion as a modular construction that applies to any topos model of single-clock guarded recursion.

1 Introduction

1.1 Synthetic guarded domain theory

Beginning with Scott and Strachey's groundbreaking investigations in 1969, the scientific study of programming semantics has been guided by the search for a topology of computation — embodied in monoidal closed categories of spaces called ***domains*** whose points can be thought of as the values of datatypes and computer programs [47,48,49,50]. The thesis of denotational semantics under Scott and Strachey is that the computational behavior of expressions in a programming language can be studied by characterizing what values they take when interpreted as continuous functions between domains; the advantage of denotational semantics over the direct/operational study of program behavior is that, unlike the latter, it is compositional and amenable to mathematical methods of reduction and abstraction.

The need to reason about increasingly complex programming languages has drawn researchers toward alternative theories of domains based on (complete, bounded, ultra-) ***metric spaces*** [37,1,46,17,8]. Metric domain theory has proved instrumental in untangling the circularities of programming language semantics involving higher-order store [9]; categories of metric spaces come equipped with a pointed endofunctor $\frac{1}{2}\cdot -$ to *scale* a given space by half, which in combination with Banach's fixed point theorem can be used to prove algebraic compactness relative to mixed-variance endofunctors whose recursive variables are guarded by $\frac{1}{2}\cdot -$. From this one obtains a relatively simple interpretation of both recursive types and programs.

Ideas from metric domain theory have been later simplified and generalized to categories of sheaves on frames with well-founded bases and the even simpler case of presheaves on well-founded posets [6], allowing for a *synthetic* and topos-theoretic approach to guarded domain theory along the lines of synthetic domain theory [28] or synthetic differential geometry [35]. The core idea of ***synthetic guarded domain theory*** (**SGDT**) is to work with a topos $\mathscr{S}$ that is equipped with an endofunctor $\blacktriangleright$ called the ***later modality***, together with a natural transformation $\mathsf{next} : \mathsf{id}_{\mathscr{S}} \longrightarrow \blacktriangleright$ and a "guarded" fixed point operator $-^{\dagger}$ ensuring that for each $f : \blacktriangleright A \longrightarrow A$, there exists a unique $f^{\dagger} : 1_{\mathscr{S}} \longrightarrow A$ such that $f \circ \mathsf{next}_A \circ f^{\dagger} = f^{\dagger}$. Under mild assumptions (*e.g.* left exactness of $\blacktriangleright$), it can be seen that $\blacktriangleright$ extends to an endofunctor on the fundamental fibration $\mathbf{P}_{\mathscr{S}} \longrightarrow \mathscr{S}$ and therefore gives rise to a true connective in the internal dependent type theory of $\mathscr{S}$.

Synthetic guarded domain theory has been employed as a metalanguage for the denotational *and* operational semantics of simple programming languages such as PCF and FPC [45,41,44]; the models of *op. cit.* can be seen as synthetic versions of Escardó's "analytic" metric model of PCF [24]. Synthetic guarded domain theory also provides the mathematical basis [11] for **Iris**, a higher-order guarded separation logic that has been used to develop operationally-based program logics for sophisticated programming languages involving higher-order store, concurrency, and a number of other computational effects [33,32].

1.2 Multi-clock guarded recursion and coinduction

Unlike both classical and ordinary synthetic domain theory, the synthetic guarded domain theory is *effective* in the sense that it gives rise to type theories satisfying the *canonicity* property [26]; this means that synthetic guarded domain theory is a *programming language* in addition to a semantic universe for denotational semantics. One early application of guarded recursion in this sense was to provide a more ergonomic and compositional method to write programs involving coinductive types or *final coalgebras*.

Consider the type of infinite streams $\mathsf{S}A$ as an example; this type is the final coalgebra for the endofunctor $\mathsf{F}_A X = A \times X$, and because F_A is ω-cocontinuous we may compute $\mathsf{S}A$ as the limit of the ω-chain $\mathsf{F}_A^n 1 \cong A^n$ by Adámek's theorem. A stream producer $\alpha : X \longrightarrow \mathsf{S}A$ must therefore decompose into a cone of finite approximations $\alpha_n : X \longrightarrow A^n$ for all $n \in \omega$; in simpler terms, we must be able to compute any finite approximation of a stream. It is not difficult to imagine programming *partial* functions on streams $\beta : \mathsf{S}A \longrightarrow \mathsf{S}B$ by general recursion; such a programming style is easily supported in languages like Haskell. But what is the appropriate linguistic construct for defining *total* functions $\beta : \mathsf{S}A \longrightarrow \mathsf{S}B$? Just as in the dual case for *inductive* data, a programming language must verify that recursive calls are justified and reject any recursive calls that would make (for instance) the projections $\beta_n : \mathsf{S}A \longrightarrow B^n$ ill-defined.

One method to ensure that recursive functions on coinductive types are total is to impose a ***syntactic guardedness*** check: every recursive call must be wrapped in a call to a constructor. Syntactic guardedness checks are employed in several type theoretic languages, such as Agda [43], Coq [22], and Idris [15,16], but they are unfortunately very brittle and not at all conducive to compositional and modular programming with higher-order functions. Type-based approaches such as ***sized types*** have been proposed as a more compositional alternative to syntactic checks [27], but the meaning of sized types as they are used in practice remains poorly understood — for instance, the version of sized types implemented in Agda is clearly inconsistent as it asserts the well-foundedness of an order with $\infty \prec \infty$; yet it remains unclear whether many sized Agda programs would survive the transition to a system in which $\infty \not\prec \infty$. [1]

A *sound* and thus more promising type-based approach to ensuring the guardedness of recursive calls arises from the later modality $\blacktriangleright$, first viewed as a programming construct by Nakano [42]. [2] The idea is to approximate the coinductive type $\mathsf{S}A \cong A \times \mathsf{S}A$ by the *guarded recursive* type $\mathsf{S}_{\blacktriangleright} A \cong A \times \blacktriangleright \mathsf{S}_{\blacktriangleright} A$:

$$\mathsf{S}_{\blacktriangleright} : \mathsf{Type} \to \mathsf{Type} \qquad (::) : A \to \blacktriangleright \mathsf{S}_{\blacktriangleright} A \to \mathsf{S}_{\blacktriangleright} A$$

The guarded fixed point operator then allows recursive definitions of functions on guarded streams, with the caveat that recursive calls must appear underneath the later modality. While this semantic / type-based restriction does automatically ensure totality, it is too conservative: we cannot, for instance, define the

[1] See `https://github.com/agda/agda/issues/2820` for a discussion of the inconsistency of Agda's sized types.
[2] The calculus of Nakano [42] was subsequently connected to metric domain theory by Birkedal, Schwinghammer, and Støvring [7].

projection functions $\mathsf{S}_{\blacktriangleright}A \to A^n$. For instance, the following attempted definition is not well-typed:

$$\begin{aligned}&\mathsf{take} : \mathbb{N} \to \mathsf{S}_{\blacktriangleright}A \to \mathsf{List}\ A\\ &\mathsf{take}\ 0\ u = []\\ &\mathsf{take}\ (n+1)\ (x :: u) = x :: \mathsf{take}\ n\ u\end{aligned}$$

1.2.1 Atkey and McBride's clock-indexed guarded recursion

At the heart of the problem discussed above is the fact that the guarded streams only approximate the coinductive streams. The remarkable suggestion of Atkey and McBride [3] is to *define* real coinductive types in terms of their guarded approximations by adding an additional notion of ***clock*** to the language; with this combination of features, arbitrary functions on coinductive types can be defined using guarded recursion. In the setting of Atkey and McBride, the later modality ensures that functions are well-defined and the clocks allow the later modality to be *removed* in a type-constrained way. The language of Atkey and McBride contains a new sort of clocks k, together with a clock-indexed family of later modalities $\blacktriangleright_k$ as well as a ***clock quantifier*** $\forall k.A[k]$. In the case where A does not depend on the clock variable k, a ***clock irrelevance*** principle is asserted stating that $A = \forall k.A$; finally the canonical map $\lambda x.\Lambda k.\mathsf{next}(x[k]) : \forall k.A[k] \longrightarrow \forall k.\blacktriangleright_k A[k]$ is asserted to have an inverse force.

A clock can be thought of metaphorically as a "time stream"; thus an element of $\forall k.\blacktriangleright_k A$ exhibits an element of $\blacktriangleright_k A$ in all time streams k; thus under this metaphor, the force operation simply instantiates this family at an *earlier* time stream to obtain an element of A. With the clock-indexed later modality in hand, it is now possible to define coinductive streams in terms of their guarded approximations by setting $\mathsf{S}A :\equiv \forall k.\mathsf{S}_{\blacktriangleright_k}A$; thus we may use guarded recursion to define the take function on coinductive streams:

$$\begin{aligned}&\mathsf{S}_{\blacktriangleright_k} : \mathsf{Type} \to \mathsf{Type}\\ &(::) : A \to \blacktriangleright_k \mathsf{S}_{\blacktriangleright_k}A \to \mathsf{S}_{\blacktriangleright_k}A\end{aligned}
\qquad
\begin{aligned}&\mathsf{uncons}_k : \mathsf{S}_{\blacktriangleright_k}A \to A \times (\blacktriangleright_k \mathsf{S}_{\blacktriangleright_k}A)\\ &\mathsf{uncons}_k\ (x :: u) = (x, u)\end{aligned}
\qquad
\begin{aligned}&\mathsf{head} : \mathsf{S}A \to A\\ &\mathsf{head}\ u = \Lambda k.\mathsf{fst}\ (\mathsf{uncons}_k\ u[k])\end{aligned}$$

$$\begin{aligned}&\mathsf{tail} : \mathsf{S}A \to \mathsf{S}A\\ &\mathsf{tail}\ u = \mathsf{force}(\Lambda k.\mathsf{snd}\ (\mathsf{uncons}_k\ u[k]))\end{aligned}
\qquad
\begin{aligned}&\mathsf{take} : \mathbb{N} \to \mathsf{S}A \to \mathsf{List}\ A\\ &\mathsf{take}\ 0\ u = []\\ &\mathsf{take}\ (\mathsf{suc}\ n)\ u = (\mathsf{head}\ u) :: (\mathsf{take}\ n\ (\mathsf{tail}\ u))\end{aligned}$$

1.2.2 Bizjak and Møgelberg's clock synchronization; Sterling and Harper's variant

One aspect of Atkey and McBride's clocks that has proved difficult to account for in a well-behaved way is that the substitution of clock variables is restricted to avoid identifying "time streams" within types: to be precise, arbitrary substitutions $[k'/k]$ are permitted in terms, but a substitution in a *type* is only permitted when it does not cause two distinct clocks to be identified. To implement this restriction, a somewhat bizarre side condition on the instantiation rule for $\forall k.A[k]$ is needed, which unfortunately appears to preclude the generalization of Atkey and McBride's clocks to the dependently typed setting, *pace* a worthy attempt by Møgelberg [40] which was thwarted by the failure of the substitution lemma.

Bizjak and Møgelberg [13] subsequently resolved these difficulties in 2015 by simply removing the restriction on clock substitution entirely: to substantiate this simplified language, they construct a *fibered* presheaf model that supports *diagonal substitutions* of clocks, which they refer to as ***clock synchronization***. With synchronization in place, there were two remaining problems left unresolved by *op. cit.*:

(i) the fibered character of the model caused coherence problems that impede the interpretation of the syntax of dependent type theory;

(ii) also missing was the ***clock irrelevance*** principle, which should at a minimum ensure that the canonical map $A \longrightarrow \forall k.A$ is an isomorphism for any type A that doesn't depend on k.

A solution to the coherence problem was found in 2017 by Sterling and Harper [52], who employed the Grothendieck construction to replace the fibered presheaf model by an equivalent *ordinary* presheaf model. Sterling and Harper also addressed clock irrelevance in two steps: first, they interpreted the clock quantifier $\forall k$ as an *intersection type* in an internal realizability model;[3] then they ensured that this intersection

[3] See also Bizjak and Birkedal [10] for a different realizability approach to multi-clock guarded recursion via generalized equilogical spaces.

is non-trivial by *forcing* the proposition $\exists k.\top$ in the ambient topos. Around the same time, Bizjak and Møgelberg [14] returned to the clock synchronization model with an analogous solution to the coherence problem and a more general approach to clock irrelevance: the quantifier $\forall k$ remains a dependent product, but they restrict the language to types that are *orthogonal* to the object of clock names. The solution of Bizjak and Møgelberg is more broadly applicable than that of Sterling and Harper because intersection need not be meaningful in an arbitrary category whereas products have a very simple universal property.

1.3 Goals and structure of this paper

Most models of single-clock synthetic guarded domain theory are given by presheaf topoi; the models of multi-clock synthetic guarded domain theory are also taken in presheaves, but of a different kind than the single-clock version. Thus the work of [3,13,14,52] on multi-clock guarded recurison raises two questions concerning the relationship between the existing models of single-clock and multi-clock guarded recursion:

(i) Does the passage from single-clock to multi-clock topos models have a universal property?

(ii) Can the multi-clock model be rephrased as a special case of the single-clock model of SGDT?

In this paper we answer both questions positively. Each multi-clock topos can be seen to be a *partial product* or *bagtopos* [55,29,30] for a certain cocartesian fibration of topoi applied at a given model of single-clock guarded recursion as hinted by Sterling [51, §2.2.6]; moreover we show that the model of synthetic guarded domain theory in the multi-clock setting is an instance of the single-clock model generalized to the *relative Grothendieck topos theory* over a given elementary topos $\mathscr{S}$. Thus we have contributed a completely modular toolkit for negotiating the two orthogonal axes of variation in multi-clock synthetic guarded domain theory: the properties of the object of clocks, and the properties of each later modality $\blacktriangleright_k$.

Structure of this paper

In **Section 2**, we introduce the topos and category theory that is needed for our technical development. As *relative* topos theory plays an important role in our work, we pay special attention to it. In **Section 3**, we define *elementary axioms* for both single-clock and multi-clock synthetic guarded domain theory in a topos, and we contribute a toolkit for constructing and transforming models of both.

(i) In **Sections 3.3 and 3.4** we show that synthetic guarded domain theory is stable under both presheaves and localization, hence any bounded geometric morphism into a topos model of SGDT lifts this model into its domain.

(ii) In **Section 3.5** we generalize the results of Birkedal *et al*. [6] by constructing models of SGDT in sheaves on frames with a well-founded basis *over an arbitrary base topos* with a natural numbers object. This generalization requires a subtle change to the definition of well-founded poset, as well as new constructive proofs of the existence of term-level guarded fixed points.

In **Section 4** we provide an explicit description of the geometric theories that extant presheaf models of single-clock guarded recursion classify. In **Section 5** we give a general construction of a multi-clock model from a single-clock model using the *bagtopoi* of Vickers [55]: in particular, a multi-clock topos classifies the theory of a "bag" or "multi-set" of points of the corresponding single-clock topos. Our characterization thus provides a geometrical universal property for multi-clock guarded recursion as a model construction on topoi, *and* as an explicit transformation of geometric theories. Finally in **Section 6**, we use these new universal properties to give a new and more abstract proof that the multi-clock topoi are in fact models of synthetic guarded domain theory in each clock.

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2 Geometric universes, topoi, and logoi

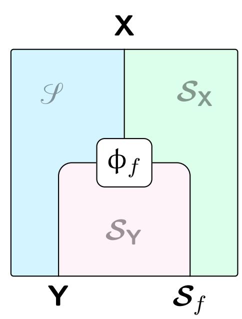
Because the topic of the present paper is logico-geometric duality in guarded domain theory, we are careful to distinguish the spatial aspects of topoi from the logical ones in both our terminology and notations. Similar conventions are employed by Vickers [56], Bunge and Funk [21], and Anel and Joyal [2].

Definition 2.1. A ***geometric universe*** [4] is a cartesian closed category $\mathscr{E}$ that has finite limits, a subobject classifier Ω, and a natural numbers object $\mathbb{N}$. A ***morphism*** $F : \mathscr{E} \longrightarrow \mathscr{F}$ of geometric universes is given by a left exact functor $\Delta_F : \mathscr{E} \longrightarrow \mathscr{F}$ equipped with a right adjoint $\Delta_F \dashv \Gamma_F : \mathscr{F} \longrightarrow \mathscr{E}$. A ***2-morphism*** $\alpha : F \Longrightarrow G$ in $[\mathscr{E}, \mathscr{F}]$ is given by a natural transformation $\Delta_F \longrightarrow \Delta_G$.

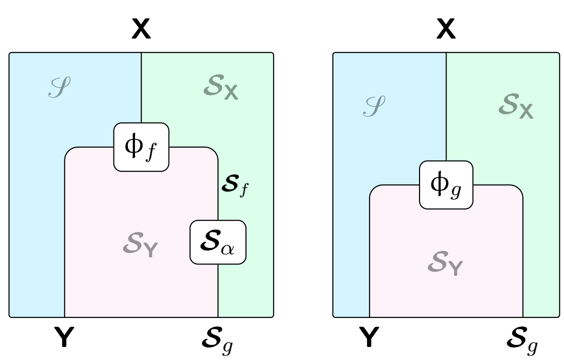
We will write **GU** for the meta-2-category of all geometric universes.

Definition 2.2. A ***left exact localization*** of a geometric universe $\mathscr{S}$ is defined to be a morphism $L : \mathscr{S} \longrightarrow \mathscr{T}$ of geometric universes such that the right adjoint functor $\Gamma_L : \mathscr{T} \longrightarrow \mathscr{S}$ is fully faithful.

Definition 2.3. A ***topos*** over a geometric universe $\mathscr{S}$ is defined to be a geometric universe $\boldsymbol{\mathcal{S}}_{\mathsf{X}}$ equipped with a structure morphism $\mathsf{X} : \mathscr{S} \longrightarrow \boldsymbol{\mathcal{S}}_{\mathsf{X}}$ of geometric universes such that the gluing fibration $\boldsymbol{\mathcal{G}}_{\mathsf{X}} = \boldsymbol{\mathcal{S}}_{\mathsf{X}} \downarrow \Delta_{\mathsf{X}} \twoheadrightarrow \mathscr{S}$ has a small separator. A ***morphism*** of $\mathscr{S}$-topoi $f : \mathsf{X} \longrightarrow \mathsf{Y}$ is then a morphism in the pseudo-coslice $\mathscr{S} \downarrow \mathbf{GU}$, *i.e.* a morphism $\boldsymbol{\mathcal{S}}_f : \boldsymbol{\mathcal{S}}_{\mathsf{Y}} \longrightarrow \boldsymbol{\mathcal{S}}_{\mathsf{X}}$ of geometric universes equipped with an isomorphism $\phi_f : \mathsf{X} \Longrightarrow \mathsf{Y}; \boldsymbol{\mathcal{S}}_f$ in $\mathbf{GU}[\mathscr{S}, \boldsymbol{\mathcal{S}}_{\mathsf{X}}]$ as depicted in the wiring diagram below:



We will write $f^* \dashv f_*$ for the adjunction $\Delta_{\boldsymbol{\mathcal{S}}_f} \dashv \Gamma_{\boldsymbol{\mathcal{S}}_f}$. A ***2-morphism*** $\alpha : f \Longrightarrow g$ in $[\mathsf{X}, \mathsf{Y}]$ is defined to be a 2-morphism $\boldsymbol{\mathcal{S}}_\alpha : \boldsymbol{\mathcal{S}}_f \Longrightarrow \boldsymbol{\mathcal{S}}_g$ such that the following wiring diagrams are equal (denote the same 2-cell):



We will write $\mathbf{Top}_{\mathscr{S}}$ for the meta-2-category of $\mathscr{S}$-topoi for a geometric universe $\mathscr{S}$. We will write $\mathbf{Log}_{\mathscr{S}} = \mathbf{Top}_{\mathscr{S}}^{\circ}$ for the meta-2-category obtained by reversing the 1-cells but not the 2-cells; we refer to an object of $\mathbf{Log}_{\mathscr{S}}$ as an $\mathscr{S}$-*logos*. Given a topos X, we may think of the dual logos $\boldsymbol{\mathcal{S}}_{\mathsf{X}}$ as the category of $\mathscr{S}$-valued *sheaves* on the space X. Indeed, the *relative* Giraud theorem states that the logos $\boldsymbol{\mathcal{S}}_{\mathsf{X}}$ can be equivalently presented as a left exact localization of a category of internal diagrams $[\mathbb{C}, \mathscr{S}]$ for some internal category $\mathbb{C}$ in $\mathscr{S}$.

[4] Geometric universes in this sense are usually referred to as *elementary topoi with natural numbers objects.*

2.1 Geometric theories and classifying topoi

The notion of *geometric theory* elucidates the relationship between topoi and logoi.

> Each geometric theory $\mathbb{T}$ determines a *classifying topos* $[\mathbb{T}]$ whose points form the category of $\mathbb{T}$-models and $\mathbb{T}$-model homomorphisms; the dual logos $\boldsymbol{\mathcal{S}}_{[\mathbb{T}]}$ is then the *classifying category* or category of contexts and substitutions for the theory $\mathbb{T}$. It is also useful to think of $\boldsymbol{\mathcal{S}}_{[\mathbb{T}]}$ as the universal extension of the geometric universe $\mathscr{S}$ by an indeterminate $\mathbb{T}$-model in the same way that the polynomial ring $A[x]$ is the free extension of a ring A by an indeterminate element.

There are several possible notions of geometric theory over $\mathscr{S}$; for the sake of this paper, we choose a particularly syntactical one. Informally a geometric theory over $\mathscr{S}$ is given by a collection of sorts σ and predicates $\overline{x : \sigma} \mid \phi(\boldsymbol{x})$, together with a collection of sequents $\overline{x : \sigma} \mid \phi(\boldsymbol{x}) \vdash \psi(\boldsymbol{x})$ in which ϕ, ψ are defined using $\left\{\bigvee_I, \bigwedge_n, \exists, \exists!, =\right\}$ where I ranges over an object of $\mathscr{S}$. To be precise, all these collections are parameterized in objects of $\mathscr{S}$, so the correct category of geometric theories arises as a *fibration* $\mathbf{Thy}_{\mathscr{S}} \longrightarrow \mathscr{S}$ whose fiber at $J \in \mathscr{S}$ is the category of J-indexed families of geometric theories with morphisms given by translations of sorts, operations, and derivable sequents. In practice, all these notions are conveniently manipulated in the internal type theory of $\mathscr{S}$.

Construction 2.4. The ***free finite product completion*** $\mathbb{C}^{\times}$ of any internal $\mathscr{S}$-category $\mathbb{C}$ can be computed explicitly in the internal language of $\mathscr{S}$ as follows:

(i) an object $\Psi \in \mathbb{C}^{\times}$ is a finite $\mathscr{S}$-cardinal[5] $|\Psi|$ together with a "type assignment" $\partial_\Psi \in \mathbb{C}^{|\Psi|}$,

(ii) a morphism $\psi : \Phi \longrightarrow \Psi$ in $\mathbb{C}^{\times}$ is given by a renaming $|\psi| : |\Psi| \longrightarrow |\Phi|$ together with a morphism $\partial_\psi : \partial_\Phi^{|\psi|} \longrightarrow \partial_\Psi$ in the product category $\mathbb{C}^{|\Psi|}$.

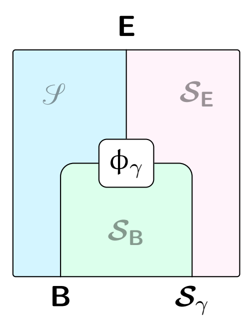
We will write $\langle - \rangle : \mathbb{C} \longrightarrow \mathbb{C}^{\times}$ for the functor sending an object to the corresponding unary product.

Example 2.5. The ***theory of an object*** ob over $\mathscr{S}$ has a single sort K, no operations, and no axioms. To be more formal, the sorts of ob are parameterized by the terminal object $1_{\mathscr{S}} \in \mathscr{S}$ and the operations and axioms are parameterized by the initial object $\emptyset_{\mathscr{S}} \in \mathscr{S}$. Letting $\mathbb{O} = \mathbf{1}^{\times}$ be the free internal $\mathscr{S}$-category with finite products generated by a single object, then $\boldsymbol{\mathcal{S}}_{[\mathsf{ob}]}$ is the category of $\mathscr{S}$-valued presheaves on $\mathbb{O}$.

Example 2.6. The ***theory of a pointed object*** el over $\mathscr{S}$ has a single sort K, a single constant $\mathsf{k} : \mathsf{K}$, and no axioms. Recalling $\mathbb{O}$ from Example 2.5, then $\boldsymbol{\mathcal{S}}_{[\mathsf{el}]}$ is the category of $\mathscr{S}$-valued presheaves on $\mathbb{O} \downarrow \langle * \rangle$.

2.2 Morphisms of topoi as relative topoi

Let $\gamma : \mathsf{E} \longrightarrow \mathsf{B}$ be a morphism of $\mathscr{S}$-topoi. In other words, we have morphisms $\mathsf{E} : \mathscr{S} \longrightarrow \boldsymbol{\mathcal{S}}_{\mathsf{E}}$, $\mathsf{B} : \mathscr{S} \longrightarrow \boldsymbol{\mathcal{S}}_{\mathsf{B}}$, $\boldsymbol{\mathcal{S}}_\gamma : \boldsymbol{\mathcal{S}}_{\mathsf{B}} \longrightarrow \boldsymbol{\mathcal{S}}_{\mathsf{E}}$, and an isomorphism $\phi_\gamma : \mathsf{E} \Longrightarrow \mathsf{B}; \boldsymbol{\mathcal{S}}_\gamma$ as in the following wiring diagram:



Forgetting the rest of the structure, the morphism $\boldsymbol{\mathcal{S}}_\gamma : \boldsymbol{\mathcal{S}}_{\mathsf{B}} \longrightarrow \boldsymbol{\mathcal{S}}_{\mathsf{E}}$ of geometric universes *also* exhibits $\boldsymbol{\mathcal{S}}_{\mathsf{E}}$ as the geometric universe underlying a $\boldsymbol{\mathcal{S}}_{\mathsf{B}}$-topos [31, Lemma B3.1.10(ii)]. In this scenario, we shall write $\mathsf{E}_\gamma : \boldsymbol{\mathcal{S}}_{\mathsf{B}} \longrightarrow \boldsymbol{\mathcal{S}}_{\mathsf{E}} = \boldsymbol{\mathcal{S}}_{\mathsf{E}_\gamma}$ for this $\boldsymbol{\mathcal{S}}_{\mathsf{B}}$-topos. This perspective allows us to take any property $\mathfrak{P}$ of topoi,

[5] To be very clear, by a finite cardinal we mean an element of the natural numbers object of $\mathscr{S}$; a morphism of finite cardinals $m \longrightarrow n$ is a function from $\mathbb{N}_{<m}$ to $\mathbb{N}_{<n}$.

e.g. local connectedness, and rephrase it as a property of *morphisms* of topoi by viewing the morphism as a topos over a different base geometric universe:

Convention 2.7 (Relative point of view)**.** A morphism $\gamma : \mathbf{E} \longrightarrow \mathbf{B}$ of $\mathscr{S}$-topoi is said to have property $\mathfrak{P}$ if $\mathbf{E}_\gamma$ has property $\mathfrak{P}$ when viewed as a $\boldsymbol{\mathcal{S}}_{\mathbf{B}}$-topos.

2.3 Internal presheaves, algebraic topoi, algebraic morphisms

Let $\mathbb{C}$ be an internal category in a geometric universe $\mathscr{S}$; we may consider the category $\mathbf{Pr}_{\mathscr{S}}\mathbb{C}$ of $\mathscr{S}$-valued presheaves on $\mathbb{C}$. The constant presheaves functor $\widehat{\mathbb{C}} : \mathscr{S} \longrightarrow \mathbf{Pr}_{\mathscr{S}}\mathbb{C}$ is then an $\mathscr{S}$-topos with $\boldsymbol{\mathcal{S}}_{\widehat{\mathbb{C}}} = \mathbf{Pr}_{\mathscr{S}}\mathbb{C}$; following the terminology of Vickers [56] and Johnstone [29] we will refer to any $\mathscr{S}$-topos equivalent to one of the form $\widehat{\mathbb{C}}$ as a *algebraic $\mathscr{S}$-topos.*[6] Thus:

Definition 2.8 (Algebraic topos)**.** An $\mathscr{S}$-topos $\mathbf{X}$ is called *algebraic* when there exists an internal category $\mathbb{C} \in \mathscr{S}$ and an equivalence of $\mathscr{S}$-topoi $\mathbf{X} \longrightarrow \widehat{\mathbb{C}}$.

Employing Convention 2.7 we generalize Definition 2.8 to morphisms of topoi.[7]

Definition 2.9 (Algebraic morphism)**.** A morphism $\beta : \mathbf{E} \longrightarrow \mathbf{B}$ of $\mathscr{S}$-topoi is likewise called *algebraic* when $\mathbf{E}_\beta$ is an algebraic $\boldsymbol{\mathcal{S}}_{\mathbf{B}}$-topos, *i.e.* there exists an internal category $\mathbb{C} \in \boldsymbol{\mathcal{S}}_{\mathbf{B}}$ such that there exists an equivalence of $\boldsymbol{\mathcal{S}}_{\mathbf{B}}$-topoi $\mathbf{E}_\beta \longrightarrow \widehat{\mathbb{C}}$.

In the scenario of Definition 2.9, we will speak of $\beta : \mathbf{E} \longrightarrow \mathbf{B}$ as the algebraic morphism presented by the $\boldsymbol{\mathcal{S}}_{\mathbf{B}}$-category $\mathbb{C}$.

Observation 2.10 (Johnstone [31, Lemma B2.5.3])**.** *If $\mathbf{E} \longrightarrow \mathbf{B}$ is the algebraic morphism presented by an internal category $\mathbb{E} \in \boldsymbol{\mathcal{S}}_{\mathbf{B}}$ and $\mathbf{F} \longrightarrow \mathbf{E}$ is the algebraic morphism presented by an internal category $\mathbb{O} \in \boldsymbol{\mathcal{S}}_{\mathbf{E}} \simeq \mathbf{Pr}_{\boldsymbol{\mathcal{S}}_{\mathbf{B}}}\mathbb{E}$, then the composite $\mathbf{F} \longrightarrow \mathbf{B}$ is the algebraic morphism presented by the internal category $\mathbb{E} \rtimes \mathbb{O} \in \boldsymbol{\mathcal{S}}_{\mathbf{B}}$ whose objects are given by pairs (u, v) with $u \in \mathbb{E}$ and $v \in \mathbb{O}u$ such that a morphism $(u, v) \longrightarrow (u', v')$ is given by a pair (f, g) where $f : u \longrightarrow u'$ and $g : v \longrightarrow f^*v'$.*

Thus algebraic morphisms are closed under composition, and moreover, composition of algebraic morphisms corresponds to the *Grothendieck construction* for the internal categories that determine them.

2.3.1 Slices and étale morphisms

Let $\mathscr{S}$ be a geometric universe and fix an object $A \in \mathscr{S}$. It is sometimes referred to as the "fundamental theorem of topos theory" that the slice $\mathscr{S} \downarrow A$ is again a geometric universe; moreover, the pullback functor $A^* : \mathscr{S} \longrightarrow \mathscr{S} \downarrow A$ can be seen to be the left adjoint part of a morphism of geometric universes $[A] = (A^* \dashv A_*)$. Thus $[A]$ can be viewed as a *discrete* $\mathscr{S}$-topos where $\boldsymbol{\mathcal{S}}_{[A]} = \mathscr{S} \downarrow A$ such that the points of the $\mathscr{S}$-valued topos $[A]$ are exactly the elements of A. Such a topos is usually referred to as *étale*:

Definition 2.11. An $\mathscr{S}$-topos $\mathbf{A}$ is called *étale* when there exists an object $A \in \mathscr{S}$ and an equivalence of $\mathscr{S}$-topoi $\mathbf{A} \longrightarrow [A]$.

Via Convention 2.7 we generalize the notion of étale $\mathscr{S}$-topos to morphisms between $\mathscr{S}$-topoi:

Definition 2.12. A morphism $p : \mathbf{E} \longrightarrow \mathbf{B}$ of $\mathscr{S}$-topoi is called *étale* when the $\boldsymbol{\mathcal{S}}_{\mathbf{B}}$-topos $\mathbf{E}_p$ is étale, *i.e.* there exists a sheaf $B \in \boldsymbol{\mathcal{S}}_{\mathbf{B}}$ together with an equivalence of $\boldsymbol{\mathcal{S}}_{\mathbf{B}}$-topoi $\mathbf{E}_p \longrightarrow [B]$.

As any object $A \in \mathscr{S}$ determines a discrete internal $\mathscr{S}$-category $\mathsf{el}\,A$, it is not difficult to see that any étale $\mathscr{S}$-topos is *also* algebraic in a canonical way: we have $[A] = \widehat{\mathsf{el}\,A}$. There is furthermore an analogue to Observation 2.10 concerning the composition of étale morphisms of topoi:

[6] The "algebraic topos" terminology tracks an precise analogy with algebraic dcpos discussed by Vickers [56].
[7] Not to be confused with the convention of referring to the inverse image $f^* : \boldsymbol{\mathcal{S}}_{\mathbf{Y}} \longrightarrow \boldsymbol{\mathcal{S}}_{\mathbf{X}}$ of a morphism of topoi $f : \mathbf{X} \longrightarrow \mathbf{Y}$ as an "algebraic morphism", which we do not employ in this paper.

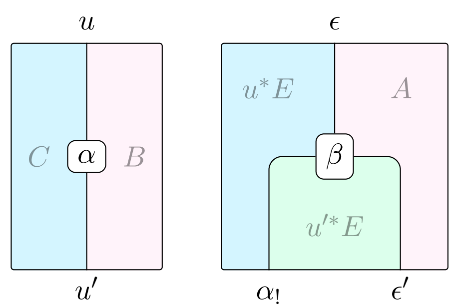
Observation 2.13. *If $\mathsf{E} \longrightarrow \mathsf{B}$ is the étale morphism presented by a sheaf $E \in \boldsymbol{\mathcal{S}}_{\mathsf{B}}$ and $\mathsf{F} \longrightarrow \mathsf{E}$ is the étale morphism presented by a sheaf $F \in \boldsymbol{\mathcal{S}}_{\mathsf{E}} \simeq \boldsymbol{\mathcal{S}}_{\mathsf{B}} \downarrow E$, then the composite $\mathsf{F} \longrightarrow \mathsf{B}$ is the étale morphism presented by the dependent sum $\sum_E F \in \boldsymbol{\mathcal{S}}_{\mathsf{B}}$.*

2.4 Partial products of topoi

We recall the theory of partial products from Johnstone [31]. Let $\mathscr{K}$ be a finitely complete 2-category and let $p : E \longrightarrow B$ be an cocartesian fibration in $\mathscr{K}$ in the sense of [31, Definition B4.4.1]. [8] Given an object $A \in \mathscr{K}$, a *partial product cone* over (p, A) at stage $C \in \mathscr{K}$ is defined to be a diagram of the following form in $\mathscr{K}$, which we write as a pair (u, ϵ):

$$\begin{tikzcd} A & u^*E \arrow[l, "\epsilon"'] \arrow[r, "q"] \arrow[d, "u^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ & C \arrow[r, "u"'] & B \end{tikzcd}$$

Johnstone defines a morphism between partial product cones $(u, \epsilon) \longrightarrow (u', \epsilon')$ to be a pair (α, β) of 2-cells as depicted below, where $\alpha_! : u^*E \longrightarrow u'^*E$ is the 1-cell assigning cocartesian lifts along $\alpha : u \Longrightarrow u'$:



We have a *2-fibration* [18] of partial product cones $\mathbf{PPCone}(p, A) \longrightarrow \mathscr{K}$ whose fiber at each $C \in \mathscr{K}$ is the category of partial product cones for (p, A) with vertex C as defined above. Then the partial product of (p, A) is defined to be a representing object $\boldsymbol{\mathcal{P}}_p A \in \mathscr{K}$ for the fibration $\mathbf{PPCone}(p, A) \longrightarrow \mathscr{K}$, if it exists. In other words, for any $C \in \mathscr{K}$ the category of morphisms $C \longrightarrow \boldsymbol{\mathcal{P}}_p A$ is equivalent to the category of partial product cones for (p, A) with vertex C. Johnstone [31] points out that the partial product can be written in the "internal language" of the 2-category $\mathscr{K}$ as the polynomial expression $\boldsymbol{\mathcal{P}}_p A = \sum_{b:B} \prod_{e:p[b]} A$.

3 Elementary synthetic guarded domain theory

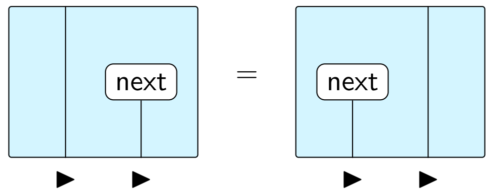
In this section, we set down *elementary* axioms for synthetic guarded domain theory in a geometric universe $\mathscr{S}$. In Sections 3.3 and 3.4 we study the stability of these axioms under the two fundamental topos-theoretic constructions: presheaves and left exact localization. In Section 3.5 we generalize the results of Birkedal *et al.* to construct a base model from a frame with a well-founded basis over any base geometric universe.

3.1 Elementary axioms for synthetic guarded domain theory

Definition 3.1. A ***later modality structure*** on $\mathscr{S}$ is given by a left exact endofunctor $\blacktriangleright : \mathscr{S} \longrightarrow \mathscr{S}$ called the *later modality* together with a natural transformation $\mathsf{next} : \mathsf{id}_{\mathscr{S}} \longrightarrow \blacktriangleright$.

[8] *Op. cit.* refers to these as *opfibrations.*

Definition 3.2. Following the terminology of Kelly [34] we refer to a later modality structure as ***well-pointed*** when the following identity of wiring diagrams holds:



Scholium 3.3. When the later modality has a left adjoint $\blacktriangleleft \dashv \blacktriangleright$, the well-pointedness condition of Definition 3.2 is equivalent to the *tick irrelevance* property isolated by Mannaa, Møgelberg, and Veltri [38].

Remark 3.4. As the later modality is left exact, it internalizes as a modality $\blacktriangleright : \Omega \longrightarrow \Omega$ on the subobject classifier that preserves finite conjunctions. Likewise, the later modality internalizes as a *fibered endofunctor* on the fundamental fibered category of $\mathscr{S}$ as in Birkedal *et al.* [6], and can hence be used informally in the internal dependent type theory of $\mathscr{S}$.

Definition 3.5. A later modality structure is said to support ***Löb induction*** when the sequent $\phi : \Omega \mid \blacktriangleright \phi \Rightarrow \phi \vdash \phi$ holds in the internal logic of $\mathscr{S}$. It is said to support ***guarded recursive terms*** when for any morphism $f : \blacktriangleright A \longrightarrow A$ there exists a unique element $f^\dagger : 1_{\mathscr{S}} \longrightarrow A$ such that $f^\dagger = f^\dagger; \mathsf{next}_A; f$.

The following result follows from the *unique choice* principle valid in any geometric universe.

Lemma 3.6. *A well-pointed later modality structure supports guarded recursive terms if and only if it supports Löb induction.*

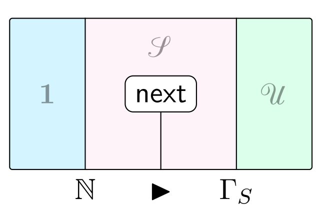
Scholium 3.7. As a consequence of Lemma 3.6, it is rarely necessary to verify the guarded recursive terms property, which is always more complex to check than Löb induction. In fact, Lemma 3.6 is an important ingredient to our constructivization of the results of Birkedal *et al.* [6] in Section 3.5: it allows us to sidestep the very technical and non-constructive proof of Lemma 8.13 in *op. cit.*

We do not here consider algebraic compactness conditions on $\mathscr{S}$ with respect to contractive functors, although these usually play an important role in the solution of domain equations in synthetic guarded domain theory [6]. Instead we take the point of view of Birkedal and Møgelberg [5] and advocate solving domain equations internally to $\mathscr{S}$ using term-level guarded recursion on universe objects. Thus we adopt the following elementary definition of a model of synthetic guarded domain theory.

Definition 3.8. An ***elementary geometric model*** of synthetic guarded domain theory is given by a geometric universe $\mathscr{S}$ equipped with a well-pointed later modality $(\blacktriangleright, \mathsf{next})$ that supports Löb induction.

Every geometric universe $\mathscr{S}$ carries a trivial model of synthetic guarded domain theory where $\blacktriangleright A = 1_{\mathscr{S}}$. It is therefore important to distinguish non-trivial models in order to implement *adequate* denotational semantics; the following definition is one possible restraint on the later modality:

Definition 3.9. Let $\mathscr{S}$ be a geometric universe equipped with a later modality structure $(\blacktriangleright, \mathsf{next})$, and let $S : \mathscr{U} \longrightarrow \mathscr{S}$ be a morphism of geometric universes. We say that $(\blacktriangleright, \mathsf{next})$ is ***globally adequate*** relative to $S : \mathscr{U} \longrightarrow \mathscr{S}$ when the following 2-cell is an isomorphism, *i.e.* we have a canonical isomorphism $\mathsf{force} : \Gamma_S \blacktriangleright \mathbb{N} \cong \Gamma_S \mathbb{N}$ where $\mathbb{N}$ is the natural numbers object of $\mathscr{S}$:



3.2 Elementary axioms for multi-clock synthetic guarded domain theory

The multi-clock variants of synthetic guarded domain theory are also accommodated under Definition 3.8; indeed, we may define an elementary geometric model of ***multi-clock synthetic guarded domain theory*** to be a geometric universe $\mathscr{S}$ equipped with an object $K \in \mathscr{S}$ and an elementary geometric model of synthetic guarded domain theory in the slice $\mathscr{S} \downarrow K$. In this scenario, K is the object of clocks and clock quantification is implemented by the dependent product functor $K_* : \mathscr{S} \downarrow K \longrightarrow \mathscr{S}$.

3.3 Stability under presheaves

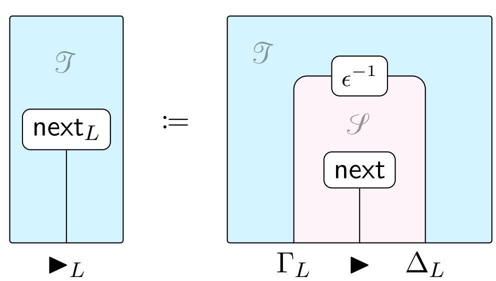
Let $\mathbb{C}$ be an internal category in a geometric universe $\mathscr{S}$, *i.e.* a small category over $\mathscr{S}$. Supposing in addition that $\mathscr{S}$ is an elementary geometric model of synthetic guarded domain theory, we may define a later modality structure pointwise on the geometric universe $\boldsymbol{\mathcal{S}}_{\widehat{\mathbb{C}}}$ of internal presheaves. In particular, we define $\blacktriangleright^{\mathbb{C}} : \boldsymbol{\mathcal{S}}_{\widehat{\mathbb{C}}} \longrightarrow \boldsymbol{\mathcal{S}}_{\widehat{\mathbb{C}}}$ to take an internal presheaf E to $c \mapsto \blacktriangleright E_c$ in the internal language of $\mathscr{S}$; likewise the natural transformation $\mathsf{next}^{\mathbb{C}} : \mathsf{id}_{\boldsymbol{\mathcal{S}}_{\widehat{\mathbb{C}}}} \longrightarrow \blacktriangleright^{\mathbb{C}}$ is given pointwise. The following Lemma 3.10 is verified by rewriting in the pictorial language of wiring diagrams, using the fact that when $\mathbb{C}$ has a terminal object, the global sections functor $\Gamma_{\widehat{\mathbb{C}}} : \boldsymbol{\mathcal{S}}_{\widehat{\mathbb{C}}} \longrightarrow \mathscr{S}$ is $\mathscr{S}$-cocontinuous and hence preserves the natural numbers object.

Lemma 3.10 (Global adequacy in presheaves)**.** *If the internal category $\mathbb{C}$ has a terminal object and $(\blacktriangleright, \mathsf{next})$ is globally adequate relative to $S : \mathscr{U} \longrightarrow \mathscr{S}$, then $(\blacktriangleright^{\mathbb{C}}, \mathsf{next}^{\mathbb{C}})$ is globally adequate relative to the composite map $S; \widehat{\mathbb{C}} : \mathscr{U} \longrightarrow \boldsymbol{\mathcal{S}}_{\widehat{\mathbb{C}}}$.*

Theorem 3.11 (Stability under presheaves)**.** *The pointwise later modality structure $(\blacktriangleright^{\mathbb{C}}, \mathsf{next}^{\mathbb{C}})$ on $\boldsymbol{\mathcal{S}}_{\widehat{\mathbb{C}}}$ is well-pointed and supports Löb induction. Hence the category of diagrams $\boldsymbol{\mathcal{S}}_{\widehat{\mathbb{C}}}$ is an elementary geometric model of synthetic guarded domain theory.*

3.4 Stability under left exact localization

Construction 3.12 (Localized later modality)**.** Let $L : \mathscr{S} \longrightarrow \mathscr{T}$ be a left exact localization, and let $(\blacktriangleright, \mathsf{next})$ be a later-modality structure on $\mathscr{S}$. We may define a canonical later-modality structure $(\blacktriangleright_L, \mathsf{next}_L)$ on $\mathscr{T}$ by conjugating with the adjunction $\Delta_L \dashv \Gamma_L$. We define $\blacktriangleright_L : \mathscr{T} \longrightarrow \mathscr{T}$ to be the composite functor $\Gamma_L; \blacktriangleright; \Delta_L$ and we define $\mathsf{next}_L : \mathsf{id}_{\mathscr{T}} \longrightarrow \blacktriangleright_L$ to be the natural transformation depicted in the following wiring diagram, in which ϵ^{-1} is the inverse to the counit $\epsilon : \Gamma_L; \Delta_L \longrightarrow \mathsf{id}_{\mathscr{E}}$ of the adjunction $\Delta_L \dashv \Gamma_L$.



The following can be proved pictorially in the language of wiring diagrams; see Appendix A.2 for details.

Lemma 3.13. *If $(\blacktriangleright, \mathsf{next})$ is a well-pointed later modality structure on $\mathscr{S}$ and $L : \mathscr{S} \longrightarrow \mathscr{T}$ is a left exact localization, then the later modality structure $(\blacktriangleright_L, \mathsf{next}_L)$ defined in Construction 3.12 is well-pointed.*

Theorem 3.14 follows nearly immediately from an internal logic argument, using the closure modality associated to any left exact localization.

Theorem 3.14 (Stability under localization)**.** *If the later modality structure $(\blacktriangleright, \mathsf{next})$ on $\mathscr{S}$ supports Löb induction, then so does $(\blacktriangleright_L, \mathsf{next}_L)$ for a left exact localization $L : \mathscr{S} \longrightarrow \mathscr{T}$. Hence if $\mathscr{S}$ is an elementary geometric model of synthetic guarded domain theory, then so is any left exact localization of $\mathscr{S}$.*

By the above, we may conclude that *any* category $\mathscr{E}$ of $\mathscr{S}$-valued sheaves inherits an elementary model of synthetic guarded domain theory from the base geometric universe $\mathscr{S}$, if it is so-equipped; note this model could depend on the chosen presentation of $\mathscr{E}$ by an $\mathscr{S}$-site. Special care must be taken, as localizations need not preserve global adequacy; for example, the localization could *trivialize* the later modality in the sense of making $\blacktriangleright_L A = 1_{\mathscr{T}}$ for all $A \in \mathscr{T}$.

3.5 Base models from intuitionistic well-founded posets

In the preceeding sections we have shown stability of elementary geometric models of synthetic guarded domain theory under basic topos theoretic constructions: presheaves and localization; from these stability properties it follows that the logos presented by *any* $\mathscr{E}$-site inherits guarded recursion from an elementary geometric model $\mathscr{E}$. But how do we construct a geometric model $\mathscr{E}$ in the first place? Birkedal *et al.* [6] provide a simple recipe for constructing such models, working in the more restrictive setting of **Set**-logoi; in particular, it is verified that for any **Set**-locale $\mathbf{X}$ whose frame of opens $\mathcal{O}_{\mathbf{X}}$ has a well-founded basis, the logos $\boldsymbol{\mathcal{S}}_{\mathbf{X}}$ is an elementary geometric model of synthetic guarded domain theory.

In this section, we carry out a (non-trivial) generalization of the results of *op. cit.* to $\mathscr{S}$-locales for an arbitrary geometric universe $\mathscr{S}$. Indeed, not only are the arguments of *op. cit.* non-constructive and thus invalid over an arbitrary geometric universe: the definition of *well-founded poset* must be adjusted as well:

(i) It will not do to define $u < v$ as $\neg(v \leq u)$ unless $\mathscr{S}$ is boolean; thus the well-founded order on a poset must be an additional structure that is compatible with the original order in a certain way.

(ii) The classical definition of well-foundedness in terms of infinite descending chains is correct if and only if $\mathscr{S}$ satisfies dependent choice.

3.5.1 Basic definitions: intuitionistic well-founded posets and frames

Definition 3.15. Let $R \subseteq P \times P$ be a binary relation in $\mathscr{S}$. The ***R-accessible elements*** of P are the smallest subset $\mathsf{Acc}_R \subseteq P$ spanned by elements $u \in P$ such that for all $v \mathrel{R} u$ we have $v \in \mathsf{Acc}_R$. We say that R is ***well-founded*** when $\mathsf{Acc}_R \subseteq P$ is P itself.

Definition 3.16. Let $(\mathbb{P}, \leq)$ be a preorder in $\mathscr{S}$; we define a ***compatible well-founded relation*** on $(\mathbb{P}, \leq)$ to be a transitive binary subrelation $\prec \subseteq \leq \subseteq \mathbb{P} \times \mathbb{P}$ satisfying the following additional axioms:

(i) *Left compatibility.* If $u \leq v$ and $v \prec w$ then $u \prec w$.

(ii) *Right compatibility.* If $u \prec v$ and $v \leq w$ then $u \prec w$.

(iii) *Well-foundedness.* The relation $\prec \subseteq \mathbb{P} \times \mathbb{P}$ is well-founded.

Definition 3.17. We define an ***intuitionistic well-founded preorder*** to be a triple $(\mathbb{P}, \leq, \prec)$ in $\mathscr{S}$ such that $(\mathbb{P}, \leq)$ is a preorder and $\prec$ is a compatible well-founded relation on $(\mathbb{P}, \leq)$. Likewise we will speak of an ***intuitionistic well-founded poset*** to refer to an intuitionistic well-founded preorder for which $\leq$ satisfies anti-symmetry.

Definition 3.18 (Connected poset)**.** Let $\mathbb{P}$ poset object in $\mathscr{S}$. We say that $u, v \in \mathbb{P}$ are *comparable* if $u \leq v$ or $v \leq u$. A poset object is *connected* if for each $u, v \in \mathbb{P}$ there exists a finite sequence $u = c_0, \ldots, c_n = v$ in $\mathbb{P}$ such that c_i and c_{i+1} are comparable for each i.

Definition 3.19 (Internal frames)**.** A *frame* in $\mathscr{S}$ is defined to be a poset object that is closed under $\mathscr{S}$-joins and finite meets, such that finite meets distribute over $\mathscr{S}$-joins.

Definition 3.20 (Basis for a frame)**.** Let $\mathbb{A}$ be a frame in $\mathscr{S}$ and let $\mathbb{K} \subseteq \mathbb{A}$ be a subposet of A; then $\mathbb{K}$ is called a *basis* for $\mathbb{A}$ when every $u \in \mathbb{A}$ is the least upper bound of all the $k \in \mathbb{K}$ such that $k \leq u$.

3.5.2 A base model in sheaves on a frame with well-founded basis

Any frame $\mathbb{A}$ in $\mathscr{S}$ gives rise to an internal site; the underlying internal category is $\mathbb{A}$ itself, and a family $\{v_i \leq u\}$ is covering when $u = \bigvee_i v_i$. We will write $\widetilde{\mathbb{A}}$ for the $\mathscr{S}$-topos obtained by setting $\boldsymbol{\mathcal{S}}_{\widetilde{\mathbb{A}}}$ to be the category of $\mathscr{S}$-valued sheaves on the internal site $\mathbb{A}$, letting the structure map $\widetilde{\mathbb{A}} : \mathscr{S} \longrightarrow \boldsymbol{\mathcal{S}}_{\widetilde{\mathbb{A}}}$ take any object of $\mathscr{S}$ to its constant sheaf. Of course we have an embedding of $\mathscr{S}$-topoi $i : \widetilde{\mathbb{A}} \hookrightarrow \widehat{\mathbb{A}}$ such that the direct image i_* regards a sheaf as a presheaf and the inverse image i^* sheafifies a presheaf.

We first construct a later modality structure on the presheaf logos $\boldsymbol{\mathcal{S}}_{\widehat{\mathbb{A}}} = \mathbf{Pr}_{\mathscr{S}}\mathbb{A} = [\mathbb{A}^\circ, \mathscr{S}]$ assuming that there exists an intuitionistic well-founded preorder $(\mathbb{K}, \leq, \prec)$ that forms a basis for $\mathbb{A}$. For each $u \in \mathbb{A}$, we will write $\mathbb{K}_{\leq u} \subseteq \mathbb{K}$ for the subposet spanned by $k \in \mathbb{K}$ such that $k \leq u$ and $\mathbb{K}_{\prec u}$ for the subposet spanned by $k \in \mathbb{K}$ such that there exists $l \in \mathbb{K}_{\leq u}$ with $k \prec_{\mathbb{K}} l$. Following Birkedal *et al.* [6] we may define a *predecessor* operation on $\mathbb{A}$ as a monotone endofunction:

$$\begin{aligned} &\mathsf{p} : \mathbb{A} \longrightarrow \mathbb{A} \\ &\mathsf{p}\, u = \textstyle\bigvee_{k \in \mathbb{K}_{\prec u}} k \end{aligned}$$

The predecessor operation induces an essential morphism of $\mathscr{S}$-topoi $\widehat{\mathsf{p}} : \widehat{\mathbb{A}} \longrightarrow \widehat{\mathbb{A}}$ whose inverse image functor $\widehat{\mathsf{p}}^* : \boldsymbol{\mathcal{S}}_{\widehat{\mathbb{A}}} \longrightarrow \boldsymbol{\mathcal{S}}_{\widehat{\mathbb{A}}}$ is given by precomposition with p. We have a natural transformation $\nu : \mathsf{id}_{\boldsymbol{\mathcal{S}}_{\widehat{\mathsf{p}}}} \longrightarrow \widehat{\mathsf{p}}^*$ defined pointwise by restriction like so:

$$\begin{aligned} &\nu_E : E \longrightarrow \widehat{\mathsf{p}}^* E \\ &\nu_E^u\, e = e_{|\mathsf{p}u} \end{aligned}$$

The following result follows immediately by computation.

Lemma 3.21. *The pair $(\widehat{\mathsf{p}}^*, \nu)$ comprise a well-pointed later modality structure on $\boldsymbol{\mathcal{S}}_{\widehat{\mathbb{A}}}$.*

Note that the later modality structure on $\boldsymbol{\mathcal{S}}_{\widehat{\mathbb{A}}}$ so-defined need not support Löb induction. Using Construction 3.12 we obtain a later modality structure $(\blacktriangleright, \mathsf{next})$ on the sheaf logos $\boldsymbol{\mathcal{S}}_{\widetilde{\mathbb{A}}}$ via the localization $\boldsymbol{\mathcal{S}}_i : \boldsymbol{\mathcal{S}}_{\widehat{\mathbb{A}}} \longrightarrow \boldsymbol{\mathcal{S}}_{\widetilde{\mathbb{A}}}$, and this structure remains well-pointed by Lemma 3.13. The following result is proved using the Kripke–Joyal semantics of $\boldsymbol{\mathcal{S}}_{\widetilde{\mathbb{A}}}$ over $\mathscr{S}$; see Appendix A.3 for the details.

Theorem 3.22 (Base model)**.** *The well-pointed later modality structure $(\blacktriangleright, \mathsf{next})$ on $\boldsymbol{\mathcal{S}}_{\widetilde{\mathbb{A}}}$ supports Löb induction, hence $\boldsymbol{\mathcal{S}}_{\widetilde{\mathbb{A}}}$ is an elementary geometric model of synthetic guarded domain theory.*

4 Classifying topoi in single-clock guarded recursion

As Birkedal *et al.* [6] have pointed out, presheaves on a well-founded poset are an instance of the sheaf models of guarded recursion considered in Section 3.5. This is so because presheaves on a well-founded poset $\mathbb{P}$ are the same as sheaves on the *algebraic locale* $\widehat{\mathbb{P}}$, whose frame of opens $\boldsymbol{\mathcal{O}}_{\widehat{\mathbb{P}}}$ is the free cocompletion of $\mathbb{P}$ under all $\mathscr{S}$-joins and whose poset of points $[\mathbb{1}, \widehat{\mathbb{P}}]$ is the free *filtered* cocompletion of $\mathbb{P}^\circ$. Thus $\mathbb{P}$ will turn out to be a (well-founded) basis for $\boldsymbol{\mathcal{O}}_{\widehat{\mathbb{P}}}$.

Explication. Let $(\mathbb{P}, \leq, \prec)$ be an intuitionistic well-founded poset in $\mathscr{S}$; we may define an $\mathscr{S}$-locale $\widehat{\mathbb{P}}$ whose frame of opens $\boldsymbol{\mathcal{O}}_{\widehat{\mathbb{P}}}$ consists of all the downsets of $\mathbb{P}$ ordered by inclusion, *i.e.* the $\mathscr{S}$-poset of $\mathscr{S}$-poset homomorphisms $[\mathbb{P}^\circ, \Omega]$.[9] Then the Yoneda embedding $\mathbb{P} \hookrightarrow \boldsymbol{\mathcal{O}}_{\widehat{\mathbb{P}}}$ exhibits $\mathbb{P}$ as a basis for $\boldsymbol{\mathcal{O}}_{\widehat{\mathbb{P}}}$, hence by Theorem 3.22 we have an elementary geometric model of synthetic guarded domain theory in $\boldsymbol{\mathcal{S}}_{\widehat{\mathbb{P}}}$. Furthermore, the geometric universe $\boldsymbol{\mathcal{S}}_{\widehat{\mathbb{P}}}$ can be seen to be the category $\mathbf{Pr}_{\mathscr{S}}\mathbb{P}$ of $\mathscr{S}$-valued presheaves on the internal site $\mathbb{P}$.

Lemma 4.1. *Let $(\mathbb{P}, \leq, \prec)$ be an intuitionistic well-founded poset in $\mathscr{S}$. We will write $\mathbb{P}_\prec = \{u \mid \exists v. u \prec v\}$ for the subposet spanned by elements lying strictly below another element. If both $\mathbb{P}$ and $\mathbb{P}_\prec$ are connected, then the later modality structure on $\boldsymbol{\mathcal{S}}_{\widehat{\mathbb{P}}}$ is globally adequate relative to $\widehat{\mathbb{P}} : \mathscr{S} \longrightarrow \boldsymbol{\mathcal{S}}_{\widehat{\mathbb{P}}}$.*

Proof. We note that $\Gamma_{\widehat{\mathbb{P}}} \blacktriangleright \mathbb{N}$ may be computed as the limit $\varprojlim_{u \in \mathbb{P}} \varprojlim_{v \prec u} \mathbb{N}$, which is also the connected limit $\varprojlim_{u \in \mathbb{P}_\prec} \mathbb{N}$; hence $\Gamma_{\widehat{\mathbb{P}}} \blacktriangleright \mathbb{N} \cong \mathbb{N} \cong \varprojlim_{u \in \mathbb{P}} \mathbb{N} \cong \Gamma_{\widehat{\mathbb{P}}} \mathbb{N}$. $\square$

[9] The frame $\boldsymbol{\mathcal{O}}_{\widehat{\mathbb{P}}}$ was referred to by Birkedal *et al.* [6] as the *ideal completion* of $\mathbb{P}$. When $\mathbb{P}$ is a total order, $\boldsymbol{\mathcal{O}}_{\widehat{\mathbb{P}}}$ is indeed the ideal completion of $\mathbb{P}$ but this need not be the case otherwise; on the other hand, the poset of points of the locale $\widehat{\mathbb{P}}$ under the specialization order is indeed the ideal completion of $\mathbb{P}^\circ$ by Diaconescu's theorem [23].

Most models of guarded recursion used in practice are indeed of this kind; in addition to the global adequacy result (Lemma 4.1) and the simplicity of working with presheaves, another advantage of the presheaf models is that they can be characterized as classifying topoi for remarkably simple and elegant geometric theories.

Lemma 4.2. *For any poset* $(\mathbb{P}, \leq)$ *in* $\mathscr{S}$*, the algebraic* $\mathscr{S}$*-topos* $\widehat{\mathbb{P}}$ *classifies the geometric theory of* filters *on* $\mathbb{P}$*,* i.e. *the theory* $\mathsf{filter}(\mathbb{P})$ *axiomatized below:*

$$\frac{u \in \mathbb{P}}{\cdot \vdash \langle u \rangle : \mathsf{prop}} \qquad \frac{u \leq v \in \mathbb{P}}{\cdot \mid \langle u \rangle \vdash \langle v \rangle} \qquad \frac{}{\cdot \mid \top \vdash \bigvee_{u \in \mathbb{P}} \langle u \rangle} \qquad \frac{u, v \in \mathbb{P}}{\cdot \mid \langle u \rangle \wedge \langle v \rangle \vdash \bigvee_{w \in \mathbb{P}, w \leq u, w \leq v} \langle w \rangle}$$

Proof. By Diaconescu's theorem [23], the topos $\widehat{\mathbb{P}}$ classifies the theory of (internally) $\mathscr{S}$-valued flat functors on $\mathbb{P}$. A flat functor a poset is the same as a filter on that poset. ☐

Example 4.3 (The topos of trees)**.** The *topos of trees* [6] over a geometric universe $\mathscr{S}$ is defined to be the algebraic topos $\widehat{\omega}$ where ω is the natural numbers object of $\mathscr{S}$ with its usual order; this is the "standard" model of synthetic guarded domain theory. We recall that $\widehat{\omega} = [\mathsf{filter}(\omega)]$ from Lemma 4.2; because $(\omega, \leq)$ is a total order in $\mathscr{S}$, the downward-directedness axiom for filters can be dropped and we see that $\widehat{\omega}$ classifies models of the following even simpler geometric theory:

$$\frac{n : \omega}{\cdot \mid \langle n \rangle : \mathsf{prop}} \qquad \frac{m \leq n : \omega}{\cdot \mid \langle m \rangle \vdash \langle n \rangle} \qquad \frac{}{\cdot \mid \top \vdash \bigvee_{i \in \omega} \langle i \rangle}$$

Example 4.4 (Successor ordinals)**.** We may consider the successor $\omega^+ = \omega \star \{\infty\}$ that adjoins a terminal element to ω. The algebraic topos $\widehat{\omega^+}$ classifies a geometric theory analogous to that of Example 4.3, but it can also be seen to classify a simpler *cartesian* theory by virtue of the fact that ω^+ has all finite meets:

$$\frac{\alpha : \omega^+}{\cdot \mid \langle \alpha \rangle : \mathsf{prop}} \qquad \frac{\alpha \leq \beta : \omega^+}{\cdot \mid \langle \alpha \rangle \vdash \langle \beta \rangle} \qquad \frac{}{\cdot \mid \top \vdash \langle \infty \rangle}$$

Lemma 4.5. *Let* $(\mathbb{P}, \sqsubseteq, \sqsubset)$ *be an intuitionistic well-founded preorder and let* $q : (\mathbb{P}, \sqsubseteq) \longrightarrow (\mathbb{P}', \leq)$ *be its poset reflection. We may define a compatible well-founded order* $\prec \subseteq \leq$ *on* $\mathbb{P}'$ *by considering the image of* $\sqsubset$ *in* $\mathbb{P}'$*,* i.e. $u \prec v \iff \forall x, y. u = qx \to v = qy \to x \sqsubset y$*.*

Example 4.6 (Well-founded trees and the plump ordering)**.** Let $p : E \longrightarrow B$ be a morphism in $\mathscr{S}$ and consider the corresponding polynomial endofunctor $\mathrm{P}_p : \mathscr{S} \longrightarrow \mathscr{S}$ taking $X \in \mathscr{S}$ to $\mathrm{P}_p X = \sum_{b:B} \prod_{e:p[b]} X$. As $\mathscr{S}$ supports W-types [39, Proposition 3.6], we may form the initial algebra $\sigma : \mathrm{P}_p \mathrm{W}_p \longrightarrow \mathrm{W}_p$ of this polynomial endofunctor. Following Fiore, Pitts, and Steenkamp [25, Example 5.4] we may equip W_p as an intuitionistic well-founded preorder, letting $(\sqsubseteq, \sqsubset)$ be the smallest relations closed under the following:

$$(\forall x : p[a]. cx \sqsubset w) \to \sigma(a, c) \sqsubseteq w \qquad (\exists x : p[a]. w \sqsubseteq cx) \to w \sqsubset \sigma(a, c)$$

The order above is adapted from Taylor [53] and called the *plump ordering* on W_p. By Lemma 4.5 we have a weakly equivalent intuitionistic well-founded poset $(\mathrm{W}_p / \sim, \leq, \prec)$, and thus the classifying topos of filters on this poset carries a model of SGDT. It is not difficult to see by unrolling definitions that Example 4.3 is the special case of this construction for the family $p : E \longrightarrow 2$ whose fibers are $\emptyset_{\mathscr{S}}$ and $1_{\mathscr{S}}$.

5 The universal property of multi-clock guarded recursion

5.1 Bagtopoi as partial products

Given a geometric theory $\mathbb{T}$ over $\mathscr{S}$, there exists a geometric theory $\mathsf{bag}(\mathbb{T})$ over $\mathscr{S}$ of families of $\mathbb{T}$-models indexed by an object of $\mathscr{S}$. We will give an explicit description of $\mathsf{bag}(\mathbb{T})$ for $\mathbb{T}$ a propositional geometric theory (*i.e.* a theory with no sorts and only nullary predicates).

Definition 5.1 (Johnstone [29])**.** Let $\mathbb{T}$ be a propositional geometric theory. Then $\mathsf{bag}(\mathbb{T})$ is the theory with a single sort K together with:

(i) for every proposition symbol ϕ of $\mathbb{T}$, a predicate $k : \mathsf{K} \mid \phi[k] : \mathsf{prop}$;

(ii) for every axiom $\cdot \mid \phi \vdash \psi$ of $\mathbb{T}$, an axiom $k : \mathsf{K} \mid \phi[k] \vdash \psi[k]$.

Definition 5.2 (Johnstone [31, Proposition B4.4.16])**.** Let $\mathbf{E}$ be an $\mathscr{S}$-topos. Let $p : [\mathsf{el}] \longrightarrow [\mathsf{ob}]$ be the generic étale morphism of $\mathscr{S}$-topoi projecting the underlying object from a pointed object. The ***(lower) bagtopos*** over $\mathbf{E}$ is the partial product $\boldsymbol{\mathcal{B}}_{\mathsf{L}}\mathbf{E} := \boldsymbol{\mathcal{P}}_p\mathbf{E}$.

Observation 5.3 (Johnstone [31, Proposition B4.4.16])**.** *For a geometric theory $\mathbb{T}$, the bagtopos $\boldsymbol{\mathcal{B}}_{\mathsf{L}}[\mathbb{T}]$ is the classifying $\mathscr{S}$-topos for the theory $\mathsf{bag}(\mathbb{T})$,* i.e. *we have $\boldsymbol{\mathcal{B}}_{\mathsf{L}}[\mathbb{T}] \simeq [\mathsf{bag}(\mathbb{T})]$.*

5.2 A universal property for Bizjak and Møgelberg's clocks

Let $\mathbb{K}$ be the $\mathscr{S}$-category having as its objects pairs $(U, f \in \omega^U)$ with U a finite $\mathscr{S}$-cardinal, while morphisms $(U, f) \to (V, g)$ are given by functions $h : U \to V$ such that $h; g \leq f \in \omega^U$ in the pointwise ordering. The internal category $\mathbb{K}$ above is exactly the *category of time objects* written $\mathbb{T}$ by Bizjak and Møgelberg [14], relativized over an arbitrary geometric universe. It was left unmentioned by *op. cit.* that $\mathbb{K}$ is the free finite coproduct completion of ω°, an observation that sets the stage for our present results.

We observe that $\mathbb{K}^\circ$ is the free finite product completion $\omega^\times$ of ω; thus we define the ***Bizjak–Møgelberg topos*** over $\mathscr{S}$ to be the algebraic $\mathscr{S}$-topos $\mathbf{BM} := \widehat{\omega^\times}$ whose total geometric universe is $\boldsymbol{\mathcal{S}}_{\mathbf{BM}} = [\mathbb{K}, \mathscr{S}]$.

Lemma 5.4 (Johnstone [31, Example B4.4.17])**.** *If $\mathbf{E}$ is the algebraic $\mathscr{S}$-topos presented by an $\mathscr{S}$-category $\mathbb{C}$, then the bagtopos $\boldsymbol{\mathcal{B}}_{\mathsf{L}}\mathbf{E}$ admits an algebraic presentation by $\mathbb{C}^\times$; in other words, we have $\boldsymbol{\mathcal{B}}_{\mathsf{L}}\widehat{\mathbb{C}} \simeq \widehat{\mathbb{C}^\times}$.*

Corollary 5.5 (Universal property)**.** *The Bizjak–Møgelberg topos $\mathbf{BM}$ is equivalent to the bagtopos $\boldsymbol{\mathcal{B}}_{\mathsf{L}}\widehat{\omega}$, hence $\mathbf{BM}$ classifies the geometric theory $\mathsf{bag}(\mathsf{filter}(\omega))$. In other words, the category of morphisms of topoi $\mathbf{X} \longrightarrow \mathbf{BM}$ is exactly the category of pairs (K, ϕ) where $K \in \boldsymbol{\mathcal{S}}_{\mathbf{X}}$ and $\phi \subseteq K \times \omega$ is an K-indexed family of filters on ω internal to $\boldsymbol{\mathcal{S}}_{\mathbf{X}}$.*

5.3 A universal property for Sterling and Harper's clocks

Let $\circleddash$ be the $\mathscr{S}$-category having as its objects pairs $(U, f \in \omega^U)$ with U a finite, nonzero $\mathscr{S}$-cardinal, while morphisms $(U, f) \to (V, g)$ are given by functions $h : U \to V$ such that $h; f \leq g \in \omega^U$ in the pointwise ordering. Observe how $\circleddash$ is exactly the category of clocks described by Sterling and Harper [52], relativized over an arbitrary geometric universe.

Remark 5.6 (Image factorization [2, 3.2.11–12])**.** Each morphism of topoi $f : \mathbf{X} \longrightarrow \mathbf{Y}$ can be factored in a composition of a surjection followed by an embedding. If moreover f is étale, then the components of its factorization are also étale.

Example 5.7. The *theory of an inhabited object* inh over $\mathscr{S}$ has a single sort K together with the axiom $\vdash \exists k : \mathsf{K}.\top$. Let $\mathbb{C}$ be an internal $\mathscr{S}$-category, and let $\mathbb{C}^\times_{inh}$ be the full subcategory of $\mathbb{C}^\times$ whose objects are nonzero cardinals. Then $\boldsymbol{\mathcal{S}}_{[\mathsf{inh}]}$ is the category of $\mathscr{S}$-valued presheaves on $\mathbf{1}^\times_{inh}$.

Definition 5.8. Let $\mathbb{T}$ be a geometric theory. Then $\mathsf{bag}_{inh}(\mathbb{T})$ is obtained by extending $\mathsf{bag}(\mathbb{T})$ with an additional axiom requiring that the sort K is inhabited.

Definition 5.9. Let $e : [\mathsf{el}] \longrightarrow [\mathsf{inh}]$ be the surjective part of the image factorization for the generic étale morphism of $\mathscr{S}$-topoi. Then the ***inhabited bagtopos*** of an $\mathscr{S}$-topos $\mathbf{E}$ is the partial product $\boldsymbol{\mathcal{B}}^{inh}_{\mathsf{L}}\mathbf{E} := \boldsymbol{\mathcal{P}}_e\mathbf{E}$.

Observation 5.10. *For a geometric theory $\mathbb{T}$, the inhabited bagtopos $\boldsymbol{\mathcal{B}}^{inh}_{\mathsf{L}}[\mathbb{T}]$ is the classifying $\mathscr{S}$-topos for the theory $\mathsf{bag}_{inh}(\mathbb{T})$,* i.e. *we have $\boldsymbol{\mathcal{B}}^{inh}_{\mathsf{L}}[\mathbb{T}] \simeq [\mathsf{bag}_{inh}(\mathbb{T})]$.*

We observe that $\circleddash \cong \omega^{\times}_{inh}$; thus we define the ***Sterling–Harper topos*** over $\mathscr{S}$ to be the algebraic $\mathscr{S}$-topos $\mathsf{SH} := \widehat{\omega^{\times}_{inh}}$ whose total geometric universe is $\boldsymbol{\mathcal{S}}_{\mathsf{SH}} = \mathbf{Pr}_{\mathscr{S}}(\omega^{\times}_{inh})$.

Lemma 5.11. *If* E *is the algebraic* $\mathscr{S}$*-topos presented by an* $\mathscr{S}$*-category* $\mathbb{C}$*, then the inhabited bagtopos* $\boldsymbol{\mathcal{B}}^{inh}_{\mathsf{L}}\mathsf{E}$ *admits an algebraic presentation by* $\mathbb{C}^{\times}_{inh}$*; in other words, we have* $\boldsymbol{\mathcal{B}}^{inh}_{\mathsf{L}}\widehat{\mathbb{C}} \simeq \widehat{\mathbb{C}^{\times}_{inh}}$.

Corollary 5.12 (Universal property)**.** *The Sterling–Harper topos* SH *is equivalent to the inhabited bagtopos* $\boldsymbol{\mathcal{B}}^{inh}_{\mathsf{L}}\widehat{\omega}$*, hence* SH *classifies the geometric theory* $\mathsf{bag}_{inh}(\mathsf{filter}(\omega))$*. In other words, the category of morphisms of topoi* $\mathbf{X} \longrightarrow \mathsf{SH}$ *is exactly the category of pairs* (K, ϕ) *where* $K \in \boldsymbol{\mathcal{S}}_{\mathbf{X}}$ *is inhabited and* $\phi \subseteq K \times \omega$ *is a* K*-indexed family of filters on* ω *internal to* $\boldsymbol{\mathcal{S}}_{\mathbf{X}}$*.*

6 Lifting guarded recursion to the bagtopos

In this section, we combine the relative point of view with our general stability results for models of SGDT to obtain an abstract proof that the Bizjak–Møgelberg topos BM carries a model of multi-clock guarded recursion. The results of this section carry over *mutatis mutandis* to the other variants of the bagtopos considered in the preceeding section.

Observation 6.1. *The universal property of the Bizjak–Møgelberg topos* BM *as the partial product* $\boldsymbol{\mathcal{P}}_p\widehat{\omega}$ *determines a morphism of* $\mathscr{S}$*-topoi* $\epsilon : [\mathsf{K}] \longrightarrow \widehat{\omega}$ *in the following configuration, where* $[\mathsf{K}] \longrightarrow \mathsf{BM}$ *is the étale morphism corresponding to the generic object:*

$$\begin{tikzcd}
\widehat{\omega} & {[\mathsf{K}]} \arrow[l, dashed, "\epsilon"'] \arrow[r, "\overline{\mathsf{K}}"] \arrow[d, "\mathsf{K}^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathsf{el}]} \arrow[d, "p"] \\
& \mathsf{BM} \arrow[r, "\mathsf{K}"'] & {[\mathsf{ob}]}
\end{tikzcd}$$

Recall that a point of $\widehat{\omega}$ is an ω-filter and a point of BM is an indexed family of ω-filters, and moreover a point of $[\mathsf{K}]$ is such an family equipped with a distinguished index; then the morphism $\epsilon : [\mathsf{K}] \longrightarrow \widehat{\omega}$ should be thought of as taking that distinguished index to the corresponding filter.

Lemma 6.2. *The morphism* $\epsilon : [\mathsf{K}] \longrightarrow \widehat{\omega}$ *is an* algebraic *morphism of* $\mathscr{S}$*-topoi that presents* $[\mathsf{K}]$ *as the algebraic* $\boldsymbol{\mathcal{S}}_{\widehat{\omega}}$*-topos* $\widehat{\mathbb{C}}$ *for an internal* $\boldsymbol{\mathcal{S}}_{\widehat{\omega}}$*-category* $\mathbb{C}$ *with a terminal object.*

Proof. Letting $\mathbb{1}$ be the terminal $\mathscr{S}$-topos (so $\boldsymbol{\mathcal{S}}_{\mathbb{1}} = \mathscr{S}$), we observe that the composite $[\mathsf{K}] \to \widehat{\omega} \to \mathbb{1}$ is the algebraic morphism of $\mathscr{S}$-topoi presented by the semidirect product $\omega^{\times} \rtimes \mathsf{K}$ and $\widehat{\omega}$ is the algebraic $\mathscr{S}$-topos presented by ω itself. There is a small fibration $\pi : \omega^{\times} \rtimes \mathsf{K} \longrightarrow \omega$ that projects out the "value" assigned to the generic clock, in which cartesian morphisms are those that leave everything but the value of the generic clock unchanged. As this fibration is small, there exists an internal category $\mathbb{E}$ in $\boldsymbol{\mathcal{S}}_{\widehat{\omega}}$ such that $\pi : \omega^{\times} \rtimes \mathsf{K} \longrightarrow \omega$ is its externalization; explicitly, the fiber $\mathbb{E}(n)$ for $n \in \omega$ is equivalent to the full subcategory of $\omega^{\times}$ spanned by objects of the form $\Gamma \times \langle n \rangle$. We finally deduce that $\epsilon : [\mathsf{K}] \longrightarrow \widehat{\omega}$ is the algebraic morphism of $\mathscr{S}$-topoi presented by the $\boldsymbol{\mathcal{S}}_{\widehat{\omega}}$-category $\mathbb{E}$: the inverse image functor ϵ^* takes the generic ω-filter $\mathsf{y}_{\omega}-$ to the relatively constant ω-filter $\Delta_{\widehat{\mathbb{E}}}\mathsf{y}_{\omega}- = \pi^*\mathsf{y}_{\omega}-$. ☐

Corollary 6.3. *The Bizjak–Møgelberg topos* BM *carries a globally adequate model of multi-clock synthetic guarded domain theory, parameterized in the generic indexing object* $\mathsf{K} \in \boldsymbol{\mathcal{S}}_{\mathsf{BM}}$*.*

Proof. The result follows from Lemma 6.2 via Lemma 4.1. Viewing $[\mathsf{K}]$ as an algebraic topos over $\widehat{\omega}$, we automatically have a model of synthetic guarded domain theory by Theorem 3.11. Global adequacy follows from Lemma 4.1 via Lemma 3.10. ☐

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A Proofs of results

A.1 *Guarded recursive terms* vs. *Löb induction*

Notation A.1 (Internal language)**.** When working in the internal language of an elementary geometric model $\mathscr{E}$ of synthetic guarded domain theory, we will make use of the notations of guarded dependent type theory [12]. In particular, we employ the notation of *delayed substitutions* for the dependent version of the later modality defined schematically below for a pair of dependent types $\Gamma \vdash A$ *type* and $\Gamma, x : A \vdash Bx$ *type*:

$$
\begin{tikzcd}
\Gamma, x : A \vdash Bx \arrow[r, "\mathsf{next}_B[x \leftarrow u]"] \arrow[d, "\pi_{Bx}"'] & \Gamma, u : \blacktriangleright A \vdash \blacktriangleright[x \leftarrow u]Bx \arrow[r] \arrow[d, "\pi_{\blacktriangleright[x \leftarrow u]Bx}"] \arrow[dr, phantom, very near start, "\lrcorner"] & \blacktriangleright(\Gamma, x : A \vdash Bx) \arrow[d, "\blacktriangleright\pi_{Bx}"] \\
\Gamma \vdash A \arrow[r, "\mathsf{next}_A"] \arrow[d, "\pi_A"'] & \Gamma \vdash \blacktriangleright A \arrow[r] \arrow[d, "\pi_{\blacktriangleright A}"] \arrow[dr, phantom, very near start, "\lrcorner"] & \blacktriangleright(\Gamma \vdash A) \arrow[d, "\blacktriangleright\pi_A"] \\
\Gamma \arrow[r, equal] & \Gamma \arrow[r, "\mathsf{next}_\Gamma"'] & \blacktriangleright\Gamma
\end{tikzcd}
$$

Observation A.2. *Let $\phi : A \longrightarrow \Omega$ be a predicate that holds for at most one element of A; then $\blacktriangleright \exists x : A.\phi x$ implies $\exists u : \blacktriangleright A. \blacktriangleright[x \leftarrow u]\phi x$.*

Lemma 3.6. *A well-pointed later modality structure supports guarded recursive terms if and only if it supports Löb induction.*

Proof. The only-if direction is immediate. For the converse, we will employ the principle of unique choice. In particular, we consider the predicate $x : A \mid \phi\, x$ defined like so:

$$\phi x :\equiv (x = f\,(\mathsf{next}_A\, x) \wedge \forall y. y = f\,(\mathsf{next}_A\, y) \Rightarrow y = x)$$

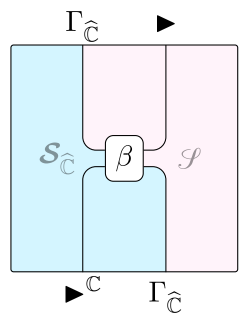
To show that there exists $x : A$ such that ϕx holds, it suffices by Löb induction to assume $\blacktriangleright \exists x : A.\phi x$ and then exhibit $x : A$ satisfying ϕx. By Observation A.2 we may assume that there exists some $u : \blacktriangleright A$ such that $\blacktriangleright[z \leftarrow u]\phi z$ holds. We choose $x :\equiv fu$ and must check that $\phi\,(fx)$ holds.

(i) *Existence.* To check that $fu = f\,(\mathsf{next}_A\,(fu))$, we will verify that $u = \mathsf{next}_A\,(fu)$ in $\blacktriangleright A$, or equivalently $\blacktriangleright[z \leftarrow u](z = fu)$. By well-pointedness, this is the same as $\blacktriangleright[z \leftarrow u](z = f\,(\mathsf{next}_A\, z))$; but $z = f\,(\mathsf{next}_A\, z)$ follows from ϕz, hence our assumption $\blacktriangleright[z \leftarrow u]\phi z$ suffices.

(ii) *Uniqueness.* Next we must check that for all $y : A$ where $y = f\,(\mathsf{next}_A\, y)$, we have $y = fu$; in other words, we must check that $f\,(\mathsf{next}_A\, y) = fu$. By congruence with f it suffices to check that $\mathsf{next}_A\, y = u$ in $\blacktriangleright A$, which (as above) is the same as to check that $\blacktriangleright[z \leftarrow u](y = z)$. This again follows from our assumption $\blacktriangleright[z \leftarrow u]\phi z$. $\square$

A.2 Stability properties for geometric models of SGDT

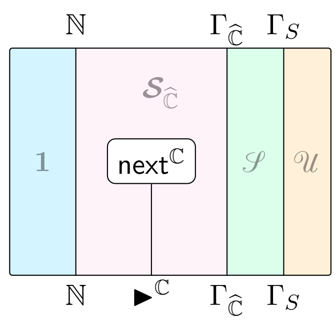
Observation A.3. *If the internal category $\mathbb{C}$ has a terminal object, then the pointwise later modality of $\boldsymbol{\mathcal{S}}_{\widehat{\mathbb{C}}}$ commutes with global sections in the sense that the following canonical 2-cell is an isomorphism:*



The 2-cell β depicted above is the distribution of $\blacktriangleright$ over the limit of a given presheaf.

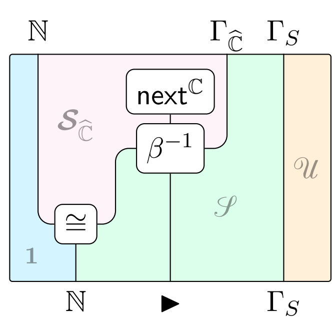
Lemma 3.10 (Global adequacy in presheaves)**.** *If the internal category $\mathbb{C}$ has a terminal object and $(\blacktriangleright, \mathsf{next})$ is globally adequate relative to $S : \mathscr{U} \longrightarrow \mathscr{S}$, then $(\blacktriangleright^{\mathbb{C}}, \mathsf{next}^{\mathbb{C}})$ is globally adequate relative to the composite map $S; \widehat{\mathbb{C}} : \mathscr{U} \longrightarrow \boldsymbol{\mathcal{S}}_{\widehat{\mathbb{C}}}$.*

Proof. We want to show that the following 2-cell is an isomorphism:



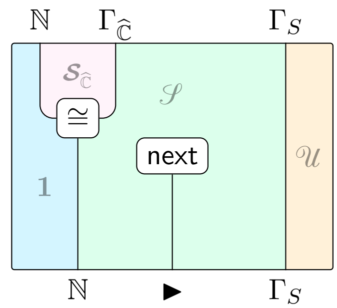
(A.1)

We may paste an isomorphism onto Diagram A.1 and check that the result is an isomorphism: the left-hand isomorphism witnesses the preservation of the natural numbers object by $\Gamma_{\widehat{\mathbb{C}}}$ and the right-hand isomorphism is from Observation A.3:



(A.2)

Diagram A.2 is equal to the following, which is an isomorphism by assumption:

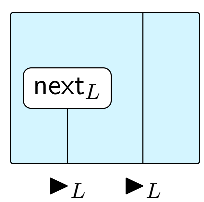


(A.3)

Thus Diagram A.1 is an isomorphism. $\square$

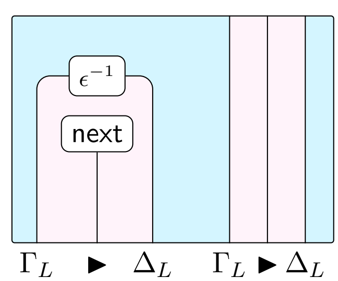
Lemma 3.13. *If* $(\blacktriangleright, \mathsf{next})$ *is a well-pointed later modality structure on* $\mathscr{S}$ *and* $L : \mathscr{S} \longrightarrow \mathscr{T}$ *is a left exact localization, then the later modality structure* $(\blacktriangleright_L, \mathsf{next}_L)$ *defined in Construction 3.12 is well-pointed.*

Proof. We proceed by rewiring in several steps.



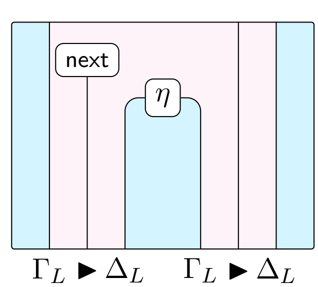
(A.4)

First we unfold definitions.



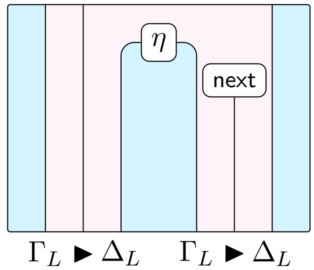
(A.5)

Next we rewire using the fact that under direct image the inverse to the counit becomes the unit.



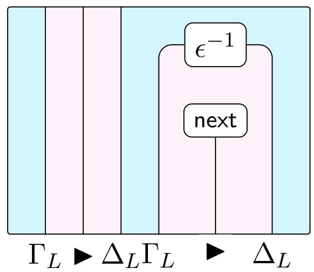
(A.6)

Then we rewire using our assumption that $(\blacktriangleright, \mathsf{next})$ is well-pointed.



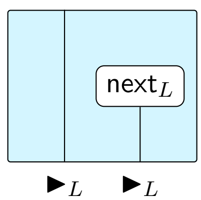
(A.7)

We rewire using the fact that the unit becomes the inverse to the counit under inverse image.



(A.8)

Folding definitions, we are done.



□

A.3 A base geometric model of SGDT

In this section, recall that $\mathbb{A}$ is a frame in a geometric universe $\mathscr{S}$ equipped with a well-founded basis $\mathbb{K}$. We will write $i : \widetilde{\mathbb{A}} \longrightarrow \widehat{\mathbb{A}}$ for the corresponding embedding of $\mathscr{S}$-topoi. We have obtained a well-pointed later modality structure $(\blacktriangleright, \mathsf{next})$ on $\boldsymbol{\mathcal{S}}_{\widetilde{\mathbb{A}}}$ from the well-pointed later modality structure $(\widehat{\mathsf{p}}^*, \nu)$ on $\boldsymbol{\mathcal{S}}_{\widehat{\mathbb{A}}}$ using Construction 3.12, but it remains to show that the former supports Löb induction.

Theorem 3.22 (Base model)**.** *The well-pointed later modality structure* $(\blacktriangleright, \mathsf{next})$ *on* $\boldsymbol{\mathcal{S}}_{\widetilde{\mathbb{A}}}$ *supports Löb induction, hence* $\boldsymbol{\mathcal{S}}_{\widetilde{\mathbb{A}}}$ *is an elementary geometric model of synthetic guarded domain theory.*

Proof. This is easily verified in the Kripke-Joyal semantics of $\boldsymbol{\mathcal{S}}_{\widetilde{\mathbb{A}}}$. Fixing $u \in \mathbb{A}$ and a closed sieve $\phi \in \Omega u$ such that $u \Vdash (\blacktriangleright \phi \Rightarrow \phi) = \top$ we must check that $u \Vdash \phi = \top$. As $\mathbb{K}$ is a basis for $\mathbb{A}$, we know that $u = \bigvee_{k \in \mathbb{K}_u} k$, so by local character it suffices to verify that $k \Vdash k^*\phi = \top$ for each $k \in \mathbb{K}_u$. By well-founded induction we may assume that $l \Vdash l^*\phi = \top$ for all $l \prec k$. From this assumption we have $k \Vdash \widehat{\mathsf{p}}^*(k^*\phi = \top)$ and hence $k \Vdash \blacktriangleright(k^*\phi = \top)$, so by our assumption that $u \Vdash (\blacktriangleright \phi \Rightarrow \phi) = \top$ we are done. □